

# Simplicity of skew group rings of abelian groups

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Given a group  $G$ , a (unital) ring  $A$  and a group homomorphism  $\sigma : G \rightarrow \text{Aut}(A)$ , one can construct the skew group ring  $A \rtimes_{\sigma} G$ . We show that a skew group ring  $A \rtimes_{\sigma} G$ , of an abelian group  $G$ , is simple if and only if its centre is a field and  $A$  is  $G$ -simple. If  $G$  is abelian and  $A$  is commutative, then  $A \rtimes_{\sigma} G$  is shown to be simple if and only if  $\sigma$  is injective and  $A$  is  $G$ -simple. As an application we show that a transformation group  $(X, G)$ , where  $X$  is a compact Hausdorff space and  $G$  is abelian, is minimal and faithful if and only if its associated skew group algebra  $C(X) \rtimes_{\sigma} G$  is simple. We also provide an example of a skew group algebra, of an (non-abelian) ICC group, for which the above conclusions fail to hold.

## 1 Introduction

Given a group  $G$ , a (unital) ring  $A$  and a group homomorphism  $\sigma : G \rightarrow \text{Aut}(A)$ , one can construct the skew group ring  $A \rtimes_{\sigma} G$  (see Section 2 for details). Skew group rings serve as an elementary way of constructing non-commutative rings. They occur naturally in different branches of mathematics, e.g. in the representation theory of Artin algebras [21], in the computation of Grothendieck groups [2], in the study of singularities [1, 3], in orbifold theory [25] and in the Galois theory of skew fields [24]. Recently, skew group rings have proven to be an important tool in the investigation of Calabi-Yau algebras derived from superpotentials [23] and in the representation theory of certain preprojective algebras [6, 9].

The ideal structure of skew group rings has been studied in depth (see e.g. [4, 5, 7, 11, 12, 17, 18, 19, 26]). Nevertheless, necessary and sufficient conditions for simplicity of a general skew group ring are not known.

The present author has shown, in his thesis [15, Theorem E.3.5] (see also [14]), that for a skew group ring  $A \rtimes_{\sigma} G$  over a commutative ring  $A$ , the subring  $A$  is a maximal

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## 1 Introduction

commutative subring of  $A \rtimes_{\sigma} G$  if and only if  $A$  has the so called *ideal intersection property* in  $A \rtimes_{\sigma} G$ , i.e. each non-zero ideal of  $A \rtimes_{\sigma} G$  intersects  $A$  non-trivially. From this one obtains the following characterization of simplicity of skew group rings over commutative rings ([15, Theorem E.6.13]).

**Theorem 1.1** ([14, 15]). *Let  $A \rtimes_{\sigma} G$  be a skew group ring where  $A$  is a commutative ring. The following two assertions are equivalent:*

- (i)  $A \rtimes_{\sigma} G$  is a simple ring;
- (ii)  $A$  is  $G$ -simple and  $A$  is a maximal commutative subring of  $A \rtimes_{\sigma} G$ .

In this article we instead turn the focus to the case when  $A$  is arbitrary, but  $G$  is abelian. Under the assumption that  $G$  is abelian and  $A$  is  $G$ -simple, we show that every non-zero ideal of  $A \rtimes_{\sigma} G$  contains a non-zero central element (Proposition 4.2). Using this we are able to give a characterization of simplicity of skew group rings of abelian groups, and this is the first main result of this article.

**Theorem 1.2.** *Let  $A \rtimes_{\sigma} G$  be a skew group ring. Consider the following three assertions:*

- (i)  $A \rtimes_{\sigma} G$  is a simple ring;
- (ii)  $A$  is  $G$ -simple and  $Z(A \rtimes_{\sigma} G)$  is a field;
- (iii)  $A$  is  $G$ -simple and  $\sigma : G \rightarrow \text{Aut}(A)$  is injective.

*The following conclusions hold:*

- (a) (i) implies (ii) and (iii);
- (b) in general (ii) and (iii) do not imply (i);
- (c) if  $G$  is an abelian group, then (i) is equivalent to (ii);
- (d) if  $G$  is an abelian group and  $A$  is a commutative ring, then (i), (ii) and (iii) are all equivalent.

In 1978 Power showed [20] that a topological dynamical system  $(X, \mathbb{Z})$  (of an infinite compact Hausdorff space  $X$ ) is minimal if and only if its associated crossed product  $C^*$ -algebra  $C^*(C(X) \rtimes \mathbb{Z})$  is simple. The present author showed in [15, Theorem E.7.6] (see also [14]) that, analogously, minimality of  $(X, \mathbb{Z})$  is equivalent to simplicity of the skew group algebra  $C(X) \rtimes \mathbb{Z}$ . Recently it was shown by de Jeu, Svensson and Tomiyama [10] that the analogous result also holds for the crossed product Banach algebra  $\ell^1(C(X) \rtimes \mathbb{Z})$ .

Let  $X$  be a compact Hausdorff space and  $G \curvearrowright X$  a strongly continuous action, inducing a group homomorphism  $\sigma : G \rightarrow \text{Aut}(C(X))$  (see Section 6 for details). This allows us to define the skew group algebra  $C(X) \rtimes_{\sigma} G$ , and as an application of Theorem 1.2 we obtain the second main result of this article which is a generalization of the aforementioned (algebraic) result on  $(X, \mathbb{Z})$ .

## 2 Preliminaries

**Theorem 1.3.** *Let  $(X, G)$  be a transformation group of a compact Hausdorff space  $X$ . Consider the following five assertions:*

- (i)  $C(X) \rtimes_{\sigma} G$  is a simple algebra;
- (ii)  $C(X)$  is  $G$ -simple and  $C(X)$  is a maximal commutative subalgebra of  $C(X) \rtimes_{\sigma} G$ ;
- (iii)  $C(X)$  is  $G$ -simple and  $Z(C(X) \rtimes_{\sigma} G)$  is a field;
- (iv)  $C(X)$  is  $G$ -simple and  $\sigma : G \rightarrow \text{Aut}(C(X))$  is injective;
- (v)  $(X, G)$  is minimal and faithful.

*The following conclusions hold:*

- (a) (i) and (ii) are equivalent and imply (iii), (iv) and (v);
- (b) (iv) and (v) are equivalent;
- (c) in general (iii), (iv) and (v) do not imply (i) and (ii);
- (d) if  $G$  is an abelian group, then (i), (ii), (iii), (iv) and (v) are all equivalent.

It is natural to ask whether this connection between minimality, faithfulness, freeness and simplicity translates to crossed product  $C^*$ -algebras. If  $(X, G)$  is a second countable locally compact transformation group with  $G$  amenable and freely acting, then it is known (see [22]) that the crossed product  $C^*$ -algebra  $C_0(X) \rtimes G$  is simple if and only if  $G$  acts minimally on  $X$ . If a group  $G$  acts on a (Borel) measurable space  $X$ , in such a way that the action is non-singular, free and ergodic, then Murray and von Neumann have shown (see e.g. [13]) that the crossed product von Neumann algebra  $L^{\infty}(X) \rtimes G$  is a factor, i.e. simple.

## 2 Preliminaries

Let  $A$  be a unital and associative ring,  $G$  a multiplicatively written group with neutral element  $e$  and  $\sigma : G \rightarrow \text{Aut}(A)$  a group homomorphism. The triple  $(A, G, \sigma)$  gives rise to a *skew group ring*, denoted  $A \rtimes_{\sigma} G$ , in the following way. Let  $\{u_g\}_{g \in G}$  be a copy of  $G$  (as a set) and define  $A \rtimes_{\sigma} G$  as the free left  $A$ -module with basis  $\{u_g\}_{g \in G}$ . Addition is defined by  $\sum_{g \in G} a_g u_g + \sum_{h \in G} b_h u_h := \sum_{g \in G} (a_g + b_g) u_g$  for  $\sum_{g \in G} a_g u_g, \sum_{h \in G} b_h u_h \in A \rtimes_{\sigma} G$ . Multiplication is defined as the bilinear extension of the rule

$$(a_g u_g)(b_h u_h) = a_g \sigma_g(b_h) u_{gh} \tag{1}$$

for  $g, h \in G$  and  $a_g, b_h \in A$ . These two operations make  $A \rtimes_{\sigma} G$  into a unital and associative ring. The multiplicative identity in  $A \rtimes_{\sigma} G$  is given by  $1_A u_e$ , but by abuse of notation we shall simply write 1. It follows from (1) that  $u_g u_{g^{-1}} = u_{g^{-1}} u_g = 1_A u_e$  and hence  $u_g^{-1} = u_{g^{-1}}$ , for  $g \in G$ . By putting  $R_g := A u_g$ , for  $g \in G$ , we see that  $A \rtimes_{\sigma} G = \bigoplus_{g \in G} R_g$  is a strongly  $G$ -graded ring. Each element  $r$  of  $A \rtimes_{\sigma} G$  may be written

### 3 Necessary conditions for simplicity of $A \rtimes_{\sigma} G$

uniquely as  $r = \sum_{g \in G} a_g u_g$  for some  $a_g \in A$ , for  $g \in G$ , of which all but finitely many are zero. The support of  $r$ , denoted  $\text{Supp}(r)$ , is defined as the finite set  $\{g \in G \mid a_g \neq 0\}$  and its cardinality will be denoted by  $|\text{Supp}(r)|$ . The multiplication rule (1) yields  $u_g a = \sigma_g(a) u_g$  for all  $g \in G, a \in A$ . This means that, for each  $g \in G$ , the map  $\sigma_g$  is implemented by the basis elements of  $A \rtimes_{\sigma} G$ , i.e.

$$\sigma_g(a) = u_g a u_g^{-1}, \quad \forall a \in A.$$

An ideal  $I$  of  $A$  is said to be  $G$ -invariant if  $\sigma_g(I) \subseteq I$  holds for all  $g \in G$ . If  $A$  and  $\{0\}$  are the only  $G$ -invariant ideals of  $A$ , then  $A$  is said to be  $G$ -simple. The fixed ring of  $A$  is defined as the set  $A^G := \{a \in A \mid \sigma_g(a) = a, \forall g \in G\}$ .

Given a subgroup  $H$  of  $G$  we denote by  $A \rtimes_{\sigma} H$  the subring of  $A \rtimes_{\sigma} G$  consisting of all elements  $r \in A \rtimes_{\sigma} G$  which satisfy  $\text{Supp}(r) \subseteq H$ . The centralizer of a subset  $S$  of a ring  $T$  will be denoted by  $C_T(S)$  and is defined as the set of all elements of  $T$  that commute with each element of  $S$ . If  $S$  is a commutative subring of  $T$  and  $C_T(S) = S$  holds, then  $S$  is said to be a *maximal commutative subring* of  $T$ . The centre of  $T$  will be denoted by  $Z(T)$ .

An automorphism  $\varphi$  of a ring  $T$  is said to be *inner* if there exists a unit  $v \in T$  such that  $\varphi(t) = v t v^{-1}$  holds for all  $t \in T$ , and *outer* otherwise. The group homomorphism  $\sigma : G \rightarrow \text{Aut}(A)$ , or simply  $G$ , is said to be *inner* if  $\sigma_g \in \text{Aut}(A)$  is inner for some  $g \in G \setminus \{e\}$ , and *outer* otherwise.

We shall make use of the following two maps of abelian groups:

$$\epsilon : A \rtimes_{\sigma} G \rightarrow A, \sum_{g \in G} a_g u_g \mapsto \sum_{g \in G} a_g; \text{ and } E : A \rtimes_{\sigma} G \rightarrow A, \sum_{g \in G} a_g u_g \mapsto a_e.$$

The map  $\epsilon$  is known as the *augmentation map* and if  $\ker(\sigma) = G$ , then  $\epsilon$  is actually a ring morphism.

### 3 Necessary conditions for simplicity of $A \rtimes_{\sigma} G$

The following proposition gives some, presumably well-known, necessary conditions for simplicity of a general skew group ring. For the sake of completeness, we include the proof.

**Proposition 3.1.** *Let  $R = A \rtimes_{\sigma} G$  be a skew group ring. If  $A \rtimes_{\sigma} G$  is simple, then the following three assertions hold:*

- (i)  $Z(A \rtimes_{\sigma} G)$  is a field;
- (ii)  $A$  is  $G$ -simple;
- (iii)  $\sigma : G \rightarrow \text{Aut}(A)$  is injective.

*Proof.* (i): Let  $a \in Z(A \rtimes_{\sigma} G) \setminus \{0\}$ . By the simplicity of  $R$  we get  $aR = Ra = R$ , which shows that  $a$  is invertible. One easily checks that  $a^{-1}$  belongs to  $Z(A \rtimes_{\sigma} G)$ .

(ii): Let  $J$  be a non-zero proper  $G$ -invariant ideal of  $A$ . Then  $J \rtimes_{\sigma} G$  is a non-zero ideal of  $A \rtimes_{\sigma} G$ . By simplicity of  $A \rtimes_{\sigma} G$  we get  $J \rtimes_{\sigma} G = A \rtimes_{\sigma} G$  and hence  $A \subseteq J \rtimes_{\sigma} G$ . Thus  $A \subseteq J$ . This shows that  $J = A$  and hence  $A$  is  $G$ -simple.

#### 4 The ideal intersection property for $Z(A \rtimes_\sigma G)$

(iii): Let  $g \in \ker(\sigma)$  be arbitrary and denote by  $I$  the two-sided ideal of  $A \rtimes_\sigma G$  generated by the element  $u_e - u_g$ . Note that for any  $s, t \in G$  and  $a_s, b_t \in A$  we get

$$a_s u_s (u_e - u_g) b_t u_t = a_s u_s b_t (u_e - u_g) u_t = a_s \sigma_s(b_t) (u_{st} - u_{sgt}). \quad (2)$$

Clearly  $\epsilon(I) = \{0\}$ . Since  $\epsilon|_A: A \rightarrow A$  is injective we conclude that  $I \cap A = \{0\}$ , which shows that  $I \subsetneq A \rtimes_\sigma G$ . By the simplicity of  $A \rtimes_\sigma G$  we conclude that  $I = \{0\}$ . In particular  $u_e - u_g = 0$  and hence  $g = e$ . This shows that  $\ker(\sigma) = \{e\}$ .  $\square$

**Remark 1.** Assertions (i)-(iii) above are in general not sufficient to guarantee simplicity of  $A \rtimes_\sigma G$  (see Example 6.1).

#### 4 The ideal intersection property for $Z(A \rtimes_\sigma G)$

The following lemma, which was inspired by [8], plays a key role in the present article.

**Lemma 4.1.** *Let  $R = A \rtimes_\sigma G$  be a skew group ring where  $G$  is abelian and  $A$  is  $G$ -simple. For each non-zero  $r \in A \rtimes_\sigma G$  there exists some  $r' \in A \rtimes_\sigma G$  with the following properties:*

- (i)  $r' \in RrR$ ;
- (ii)  $E(r') = 1$ ;
- (iii)  $|\text{Supp}(r')| \leq |\text{Supp}(r)|$ .

*Proof.* Take an arbitrary non-zero element  $r \in R$ . We can write  $r = \sum_{g \in G} a_g u_g$ , where  $a_g \in A$  is zero for all but finitely many  $g \in G$ . Since  $r$  is non-zero we can choose some  $h \in G$  such that  $a_h \neq 0$ . The element  $ru_{h^{-1}}$  clearly belongs to  $RrR$  and we note that  $|\text{Supp}(ru_{h^{-1}})| = |\text{Supp}(r)|$  and  $E(ru_{h^{-1}}) = a_h \neq 0$ . Thus, without loss of generality, we may replace  $r$  by  $ru_{h^{-1}}$  and can therefore assume that  $r = \sum_{g \in G} a_g u_g$  is such that  $a_e \neq 0$ . The set

$$J = \{E(s) \mid s \in RrR \text{ such that } \text{Supp}(s) \subseteq \text{Supp}(r)\}$$

contains the non-zero element  $a_e$  (since  $r \in RrR$ ) and hence  $J$  is a non-zero ideal of  $A$ . We claim that  $J$  is  $G$ -invariant. Indeed, if  $a \in J$ , then  $a + \sum_{g \in \text{Supp}(r) \setminus \{e\}} b_g u_g \in RrR$  for some  $b_g \in A$ ,  $g \in \text{Supp}(r) \setminus \{e\}$ . For any  $h \in G$ , we get

$$RrR \ni u_h(a + \sum_{g \in \text{Supp}(r) \setminus \{e\}} b_g u_g) u_{h^{-1}} = \sigma_h(a) + \sum_{g \in \text{Supp}(r) \setminus \{e\}} \sigma_h(b_g) \underbrace{u_{hgh^{-1}}}_{=u_g}$$

which yields  $\sigma_h(a) \in J$ . This shows that  $J$  is  $G$ -invariant. By the  $G$ -simplicity of  $A$  we conclude that  $1 \in A = J$ . Hence there is some  $r' := 1 + \sum_{g \in \text{Supp}(r) \setminus \{e\}} b_g u_g \in RrR$ , for some  $b_g \in A$ ,  $g \in \text{Supp}(r) \setminus \{e\}$ , which clearly satisfies (i)-(iii).  $\square$

Recall from [16] that a subring  $S$  of a ring  $T$  is said to have the *ideal intersection property* (in  $T$ ) if  $S \cap I \neq \{0\}$  holds for each non-zero ideal  $I$  of  $T$ .

**Proposition 4.2.** *Let  $R = A \rtimes_\sigma G$  be a skew group ring where  $G$  is abelian and  $A$  is  $G$ -simple. Every non-zero ideal of  $R$  has non-empty intersection with  $Z(R) \cap (1 + \sum_{g \in G \setminus \{e\}} Au_g)$ . In particular,  $Z(R)$  has the ideal intersection property in  $R$ .*

*Proof.* Let  $I$  be a non-zero ideal of  $R$ . Choose some  $r \in I \setminus \{0\}$  such that  $|\text{Supp}(r)|$  is as small as possible. By Lemma 4.1 there exists some  $r' \in RrR \subseteq I$  such that  $E(r') = 1$  and  $|\text{Supp}(r')| \leq |\text{Supp}(r)|$ . In fact, by minimality of  $|\text{Supp}(r)|$  among all non-zero elements of  $I$ , we have  $|\text{Supp}(r')| = |\text{Supp}(r)|$ . Let  $a \in A$  be arbitrary. Note that  $E(r'a - ar') = a - a = 0$  and thus  $|\text{Supp}(r'a - ar')| < |\text{Supp}(r')|$ . By the minimality of  $|\text{Supp}(r')|$  and the fact that  $r'a - ar' \in I$  we conclude that  $r'a - ar' = 0$ . This shows that  $r'$  belongs to the centralizer of  $A$ . Now, let  $g \in G$  be arbitrary. Note that  $E(u_g r' u_g^{-1} - r') = 1 - 1 = 0$  and thus  $|\text{Supp}(u_g r' u_g^{-1} - r')| < |\text{Supp}(r')|$ . Again, since  $u_g r' u_g^{-1} - r' \in I$ , by the minimality of  $|\text{Supp}(r')|$  we get  $u_g r' u_g^{-1} - r' = 0$ . This shows that  $u_g r' = r' u_g$ , for all  $g \in G$ . Since  $R = A \rtimes_\sigma G$  is generated as a ring by the elements of  $A$  and  $\{u_g\}_{g \in G}$ , we conclude that  $r' \in I \cap Z(R) \cap (1 + \sum_{g \in G \setminus \{e\}} Au_g)$ .  $\square$

The following lemma can sometimes be used to decide if  $Z(A \rtimes_\sigma G)$  is a field or not.

**Lemma 4.3.** *Let  $A \rtimes_\sigma G$  be a skew group ring. Consider the following assertions:*

- (i)  $Z(A \rtimes_\sigma G) \subseteq A$ ;
- (ii)  $Z(A \rtimes_\sigma G) = A^G \cap Z(A)$ ;
- (iii)  $Z(A \rtimes_\sigma G)$  is a field.

*The following conclusions hold:*

- (a) (i) and (ii) are equivalent;
- (b) if  $A$  is  $G$ -simple, then (i) and (ii) imply (iii);
- (c) if  $G$  is an orderable abelian group, then (iii) implies (i) and (ii).

*Proof.* (a) (i) $\Rightarrow$ (ii): Let  $a \in Z(A \rtimes_\sigma G) \subseteq A$ . Then  $au_g = u_g a$  holds for all  $g \in G$ . Hence  $(a - \sigma_g(a))u_g = 0$ , or equivalently  $a = \sigma_g(a)$ , for all  $g \in G$ . Hence  $Z(A \rtimes_\sigma G) \subseteq A^G \cap Z(A)$ . The other inclusion is straightforward.

(ii) $\Rightarrow$ (i): This is trivial.

(b) (ii) $\Rightarrow$ (iii): Suppose that  $A$  is  $G$ -simple. Let  $a \in A^G \cap Z(A)$  be non-zero. Then  $Aa$  is a non-zero  $G$ -invariant ideal of  $A$ . Thus  $Aa = A$ . In particular,  $1 \in Aa$ , which shows that  $a$  is invertible in  $A$  and one can easily check that the inverse of  $a$  belongs to  $A^G \cap Z(A)$ .

(c) (iii) $\Rightarrow$ (i): Suppose that  $G$  is an orderable abelian group. Assume that  $Z(A \rtimes_\sigma G) \cap Au_g \neq \{0\}$  for some  $g \in G \setminus \{e\}$  and take some non-zero  $cu_g \in Z(A \rtimes_\sigma G) \cap Au_g$ . Then  $1 + cu_g \in Z(A \rtimes_\sigma G) \setminus \{0\}$  is invertible. Using that  $G$  is an orderable group, we may without loss of generality assume that  $g > e$ . Let  $r$  be the inverse of  $1 + cu_g$  and write  $r = r_{h_1}u_{h_1} + \dots + r_{h_k}u_{h_k}$ , where  $r_{h_i} \in A \setminus \{0\}$  for some distinct  $h_1, \dots, h_k \in G$  such that  $h_1 < \dots < h_k$ . The term of lowest degree in the product  $(1 + cu_g)r$  is  $1r_{h_1}u_{h_1}$ , and the term of highest degree is  $cu_g r_{h_k}u_{h_k} = c\sigma_g(r_{h_k})u_{gh_k}$ , which is non-zero since

## 5 Injectivity of $\sigma : G \rightarrow \text{Aut}(A)$ and maximal commutativity of $A$

$cu_g$  is invertible. On the other hand,  $(1 + cu_g)r = 1$  is homogeneous and therefore  $k = 1$ . Hence  $r_{h_1}u_{h_1} + c\sigma_g(r_{h_k})u_{gh_k} = 1$ , but this is a contradiction since  $g > e$ . Hence  $Z(A \rtimes_\sigma G) \subseteq A$ .  $\square$

**Example 4.1** (Inner actions and simplicity). Let  $A = M_2(\mathbb{R})$  and  $G = \mathbb{Z}/2\mathbb{Z}$ . Put  $M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$  and note that  $M^2 = -I$ . Define  $\sigma : G \rightarrow \text{Aut}(A)$  by  $\sigma_0 = \text{id}_A$  and  $\sigma_1(a) = MaM^{-1}$  for  $a \in A$ . The action of  $G$  is clearly *inner*. We claim that  $A \rtimes_\sigma G$  is a simple ring. Since  $A$  is simple, and therefore  $G$ -simple, it follows by Theorem 1.2(c) that it is enough to show that  $Z(A \rtimes_\sigma G)$  is a field. By a straightforward calculation we get

$$Z(A \rtimes_\sigma G) = \{a_0Iu_0 + a_1Mu_1 \in A \rtimes_\sigma G \mid a_0, a_1 \in \mathbb{R}\}$$

which is a field. Indeed, let  $a_0Iu_0 + a_1Mu_1$  be an arbitrary non-zero element of  $Z(A \rtimes_\sigma G)$ . Using the fact that  $a_0^2 + a_1^2 \neq 0$  it follows by elementary linear algebra that the equation  $(a_0Iu_0 + a_1Mu_1)(b_0Iu_0 + b_1Mu_1) = 1$  always has a unique solution  $(b_0, b_1) \in \mathbb{R}^2$ .

**Remark 2.** In [5, Proposition 2.1] Crow proved the claim of Corollary 4.4 below. Example 4.1 shows that outerness of the action is not a necessary condition for simplicity of the corresponding skew group ring. It also motivates the need for Theorem 1.2(c).

We shall now show that [5, Proposition 2.1] can easily be obtained as a corollary of Theorem 1.2.

**Corollary 4.4.** *Let  $A \rtimes_\sigma G$  be a skew group ring where  $G$  is abelian and outer. The following two assertions are equivalent:*

- (i)  $A \rtimes_\sigma G$  is a simple ring;
- (ii)  $A$  is  $G$ -simple.

*Proof.* (i) $\Rightarrow$ (ii): This follows from Theorem 1.2(a).

(ii) $\Rightarrow$ (i): By Lemma 4.3(b) and Theorem 1.2(c) it is enough to show that  $Z(A \rtimes_\sigma G) \subseteq A$ . Let  $r = \sum_{g \in G} a_g u_g$  be an arbitrary non-zero element of  $Z(A \rtimes_\sigma G)$ . Take  $g \in G$  such that  $a_g \neq 0$ . Since  $r \in Z(A \rtimes_\sigma G)$  we conclude that  $a_g \in A^G$  and

$$ba_g = a_g \sigma_g(b), \quad \forall b \in A. \tag{3}$$

Using that  $a_g \in A^G$  it is clear that the set  $J := Aa_gA = Aa_g = a_gA$  is a non-zero  $G$ -invariant ideal of  $A$  and hence  $J = A$ . Thus,  $1 = a_g c$  for some  $c \in A$ . From (3) we get  $1 = \sigma_g(1) = a_g \sigma_g(c) = ca_g$ , which shows that  $a_g$  is invertible. Therefore  $\sigma_g(b) = a_g^{-1}ba_g$  for all  $b \in A$ , so  $\sigma_g$  is inner. We now conclude that  $g = e$ .  $\square$

## 5 Injectivity of $\sigma : G \rightarrow \text{Aut}(A)$ and maximal commutativity of $A$

Maximal commutativity of  $A$  in  $A \rtimes_\sigma G$  implies injectivity of  $\sigma : G \rightarrow \text{Aut}(A)$ . If  $A$  is e.g. an integral domain, then it is easy to see that the two assertions are equivalent. The same

conclusion does not, however, hold for an arbitrary commutative ring  $A$ . The following proposition describes a situation in which the two assertions are in fact equivalent.

Let  $K$  denote the kernel of the group homomorphism  $\sigma : G \rightarrow \text{Aut}(A)$ .

**Proposition 5.1.** *Let  $R = A \rtimes_{\sigma} G$  be a skew group ring where  $G$  is an abelian group and  $A$  is a commutative and  $G$ -simple ring. Then  $C_R(A) = A \rtimes_{\sigma} K$ . In particular,  $A$  is a maximal commutative subring of  $A \rtimes_{\sigma} G$  if and only if  $\sigma$  is injective.*

*Proof.* Let  $\sum_{g \in K} a_g u_g$  be an arbitrary element of  $A \rtimes_{\sigma} K$ . For any  $a \in A$  we have  $a \sum_{g \in K} a_g u_g = \sum_{g \in K} a_g \sigma_g(a) u_g = \sum_{g \in K} a_g u_g a$ . This shows that  $A \rtimes_{\sigma} K \subseteq C_R(A)$ . Now let  $\sum_{g \in G} a_g u_g \in C_R(A) \setminus \{0\}$  be arbitrary. Take  $h \in G$  such that  $a_h \neq 0$ . Note that  $a_h u_h \in C_R(A)$  since  $C_R(A)$  is  $G$ -graded. Consider the set

$$I = \{b \in A \mid (\sigma_h(a) - a)b = 0, \forall a \in A\}.$$

It is clear that  $I$  is an ideal of  $A$  and it is non-zero since  $a_h \in I$ . Take  $b \in I$  and  $g \in G$ . Then  $(\sigma_g(\sigma_h(a)) - \sigma_g(a))\sigma_g(b) = 0$  or equivalently  $(\sigma_h(\sigma_g(a)) - \sigma_g(a))\sigma_g(b) = 0$  holds for all  $a \in A$ . This shows that  $(\sigma_h(c) - c)\sigma_g(b) = 0$  holds for all  $c \in A$  and hence  $\sigma_g(b) \in I$ . Thus,  $I$  is  $G$ -invariant. By assumption we get  $I = A$ , so  $1 \in I$  which yields  $\sigma_h = \text{id}_A$ , i.e.  $h \in K$ . Since  $h$  was arbitrarily chosen we conclude that  $C_R(A) \subseteq A \rtimes_{\sigma} K$ .  $\square$

## 6 An application to topological dynamical systems

Let  $(X, G)$  be a *transformation group* consisting of a topological group  $G$  acting on a compact Hausdorff space  $X$ . Furthermore, assume that the action  $G \curvearrowright X$  is *strongly continuous*, i.e. the map  $G \times X \rightarrow X$ ,  $(g, x) \mapsto g.x$  is continuous with respect to the respective topologies.

The algebra of complex-valued continuous functions on  $X$ , where the operations of addition and multiplication are defined pointwise, is denoted by  $C(X)$ . We define  $\|f\| := \sup_{x \in X} |f(x)|$ , for  $f \in C(X)$ , and one easily checks that this defines a norm on  $C(X)$  which turns it into a unital  $C^*$ -algebra.

The transformation group  $(X, G)$  induces a group homomorphism

$$\sigma : G \rightarrow \text{Aut}(C(X)), \quad \sigma_g(f)(x) = f(g^{-1}.x), \quad g \in G, f \in C(X), x \in X. \quad (4)$$

It follows by the strong continuity of the action, that the automorphisms  $\sigma_g \in \text{Aut}(C(X))$ , for  $g \in G$ , are all continuous. We call  $C(X) \rtimes_{\sigma} G$  the *skew group algebra*<sup>1</sup> associated to the transformation group  $(X, G)$ .

**Definition 6.1.** If there, for each  $g \in G \setminus \{e\}$ , exists some  $x \in X$  such that  $g.x \neq x$ , then the transformation group  $(X, G)$  is said to be *faithful*. A subset  $V \subseteq X$  is said to be  *$G$ -invariant* if  $g.V \subseteq V$  holds for all  $g \in G$ . If  $X$  contains no non-empty proper closed  $G$ -invariant subset, then the transformation group  $(X, G)$  is said to be *minimal*.

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<sup>1</sup>The completion of this skew group algebra with respect to a suitable norm would be called a *crossed product  $C^*$ -algebra* by  $C^*$ -algebraists. In non-commutative ring theory, however, a skew group algebra is a special case of the more general (algebraic) crossed product construction.



**Remark 3.** Note that a subset  $V \subseteq X$  is  $G$ -invariant if and only if  $g.V = V$  holds for all  $g \in G$ . Minimality of  $(X, G)$  may equivalently be stated as saying that for each  $x \in X$  the orbit of  $x$ , i.e. the set  $\{g.x \mid g \in G\}$ , is a dense subset of  $X$ .

Let  $\mathcal{P}_{\text{cl}}(X)$  denote the set of all closed subsets of  $X$ , and  $\text{Ideal}_{\text{cl}}(C(X))$  denote the set of all closed ideals of  $C(X)$ . There is a one-to-one correspondence between these sets. Indeed, consider the map

$$\text{Ideal}_{\text{cl}}(C(X)) \ni I \xrightarrow{\varphi} \{x \in X \mid f(x) = 0 \text{ for all } f \in I\} \in \mathcal{P}_{\text{cl}}(X)$$

and the map

$$\mathcal{P}_{\text{cl}}(X) \ni V \xrightarrow{\psi} \{f \in C(X) \mid f|_V \equiv 0\} \in \text{Ideal}_{\text{cl}}(C(X)).$$

It follows that  $\varphi$  and  $\psi$  are well-defined and that  $\psi \circ \varphi = \text{id}_{\text{Ideal}_{\text{cl}}(C(X))}$  and  $\varphi \circ \psi = \text{id}_{\mathcal{P}_{\text{cl}}(X)}$ .

**Lemma 6.1.**  *$(X, G)$  is faithful if and only if  $\sigma$  (defined by (4)) is injective.*

*Proof.* Note that if  $|X| = 1$ , then both assertions are equivalent. Let us therefore assume that  $|X| > 1$ . If  $(X, G)$  is not faithful, then there is some  $g \in G \setminus \{e\}$  such that  $g.x = x$  for all  $x \in X$ . It then follows by (4) that  $\sigma_{g^{-1}} = \text{id}_{C(X)}$ , thus  $\sigma$  is not injective. Conversely, let  $(X, G)$  be faithful. Seeking a contradiction, suppose that  $\sigma$  is not injective. There is some  $g \in G \setminus \{e\}$  such that  $f(g^{-1}.x) = f(x)$  for all  $f \in C(X)$  and  $x \in X$ . Since  $(X, G)$  is faithful, there is some  $x \in X$  such that  $g^{-1}.x \neq x$ . By Urysohn's lemma (and the fact that  $|X| > 1$ ) we conclude that there is some  $f : X \rightarrow [0, 1] \subseteq \mathbb{C}$  such that  $f(g^{-1}.x) \neq f(x)$ . This is a contradiction.  $\square$

**Lemma 6.2.** *The following four assertions are equivalent:*

- (i)  $(X, G)$  is minimal;
- (ii) There is no non-empty closed proper  $G$ -invariant subset of  $X$ ;
- (iii)  $C(X)$  is  $G$ -simple with respect to closed ideals;
- (iv)  $C(X)$  is  $G$ -simple.

*Proof.* (i) $\Leftrightarrow$ (ii): This is indeed the definition.

(ii) $\Leftrightarrow$ (iii): Note that  $\varphi$  and  $\psi$  also give rise to a one-to-one correspondence between closed  $G$ -invariant subsets of  $X$  and closed  $G$ -invariant (with respect to  $\sigma$ ) ideals of  $C(X)$ .

(iii) $\Rightarrow$ (iv): Suppose that  $C(X)$  is  $G$ -simple with respect to closed ideals. Let  $I$  be a non-zero  $G$ -invariant ideal of  $C(X)$ . We wish to show that  $I = C(X)$ . Denote by  $\bar{I}$  the closure of  $I$ , and note that this is also an ideal of  $C(X)$ . The maps  $\sigma_g : C(X) \rightarrow C(X)$ , for  $g \in G$ , are continuous and hence the  $G$ -invariance of  $I$  implies  $\sigma_g(\bar{I}) \subseteq \bar{I}$ , for  $g \in G$ . This shows that  $\bar{I}$  is a  $G$ -invariant (and closed) ideal of  $C(X)$ . By the assumption we get  $\bar{I} = C(X)$ . Since  $C(X)$  is a unital  $C^*$ -algebra (and in particular a Banach algebra), the closure of any proper ideal is still a proper ideal. Therefore we conclude that  $I = C(X)$ .

(iv) $\Rightarrow$ (iii): This is trivial.  $\square$

### 6.1 A faithful, minimal and non-free action of an ICC group

Recall that a group  $G$  is said to be an *ICC group* if it has the *infinite conjugacy class property*, i.e. for each  $g \in G \setminus \{e\}$  the set  $\{hgh^{-1} \mid h \in G\}$  is infinite. Clearly, finite groups and abelian groups can not be ICC.

**Proposition 6.3.** *Let  $A \rtimes_\sigma G$  be a skew group ring. If  $G$  is an ICC group, then  $Z(A \rtimes_\sigma G) \subseteq A$ .*

*Proof.* Let  $r = \sum_{g \in G} a_g u_g$  be an element of  $Z(A \rtimes_\sigma G)$ . For any  $h \in G$  we have

$$\sum_{g \in G} a_g u_g = u_h \left( \sum_{g \in G} a_g u_g \right) u_h^{-1} = \sum_{g \in G} \sigma_h(a_g) u_{hgh^{-1}} = \sum_{s \in G} \sigma_h(a_{h^{-1}sh}) u_s.$$

Take  $g \in \text{Supp}(r)$  and note that  $a_g = \sigma_h(a_{h^{-1}gh}) \neq 0$  for all  $h \in G$ . Since  $G$  is an ICC group and  $\text{Supp}(r)$  is finite we get  $g = e$ . This shows that  $Z(A \rtimes_\sigma G) \subseteq A$ .  $\square$

Given a transformation group  $(X, G)$  and  $x \in X$  we let  $\text{Stab}_G(x) := \{g \in G \mid g.x = x\}$  denote the *stabilizer subgroup* of  $x$  in  $G$ .

**Lemma 6.4.** *Let  $G$  be a group which acts faithfully on a set  $X$ . If the set  $\text{Stab}_G(x).y$  is infinite for any two  $x, y \in X$  such that  $x \neq y$ , then  $G$  is an ICC group.*

*Proof.* Let  $g \in G \setminus \{e\}$ . Then there is some  $x \in X$  such that  $y := g.x \neq x$ . For any  $h \in \text{Stab}_G(x)$  we have  $hgh^{-1}.x = h.(g.x) = h.y$ . By the assumption  $\{hgh^{-1}.x \mid h \in \text{Stab}_G(x)\}$  is infinite, so in particular  $G$  is an ICC group.  $\square$

**Proposition 6.5.**  *$\text{Homeo}(S^1)$ , the group of all homeomorphisms of the circle  $S^1$ , is an ICC group.*

*Proof.* The group  $G = \text{Homeo}(S^1)$  acts on  $X = S^1$  in an obvious way and this action is clearly faithful. Let  $x, y \in S^1$  such that  $x \neq y$ . Take any  $z \in S^1$  such that  $z \neq x$ . We now define an invertible piecewise linear map  $f_z : S^1 \rightarrow S^1$  satisfying  $f_z(y) = z$  and  $f_z \in \text{Stab}_G(x)$ . This is always possible since  $z \neq x$ . We can choose  $z$  in infinitely many ways and hence  $\text{Stab}_G(x).y$  is infinite. By Lemma 6.4, the desired conclusion follows.  $\square$

**Example 6.1.** Let  $X = S^1$  be the circle,  $G = \text{Homeo}(S^1)$  the group of all homeomorphisms of  $S^1$  and consider the skew group algebra  $C(X) \rtimes_\sigma G$  where  $\sigma$  is defined by (4). It is easy to see that the action  $G \curvearrowright X$  is faithful and minimal. Hence, by Lemma 6.1  $\sigma$  is injective and by Lemma 6.2  $C(X)$  is  $G$ -simple.

By Proposition 6.5,  $G$  is an ICC group and by combining Proposition 6.3 and Lemma 4.3(b), we conclude that  $Z(C(X) \rtimes_\sigma G)$  is a field.

We claim that  $C(X) \rtimes_\sigma G$  is not simple. By Theorem 1.1 we need to show that  $C(X)$  is not a maximal commutative subalgebra of  $C(X) \rtimes_\sigma G$ . To see this, take  $g \in G \setminus \{e\}$  such that  $g^{-1}(x) = x$  for all  $x \in [0, \frac{1}{2}]$ . Choose a non-zero  $f_g \in C(X)$  such that  $f_g(x) = 0$  for all  $x \in [\frac{1}{2}, 1]$ . Then it follows that  $(\sigma_g(b) - b)f_g = 0$  for any  $b \in C(X)$ . Hence  $f_g u_g$  commutes with each element of  $C(X)$ . This shows that  $C(X)$  is not a maximal commutative subalgebra.

**Remark 4.** A minimal and faithful action of an abelian group on a compact Hausdorff space is necessarily *free*, in the sense that if  $g \in G \setminus \{e\}$  then for any  $x \in X$  we have  $g.x \neq x$ . The action in Example 6.1 is clearly non-free.

## 7 Proof of the main results

We are now fully prepared to prove the main results of this article.

*Proof of Theorem 1.2.* (a) This follows immediately from Proposition 3.1.

(b) Consider Example 6.1. In the example it is explained that (ii) and (iii) hold, but that (i) fails to hold.

(c) We need to show that (ii) implies (i). The rest follows from (a). Suppose that  $A$  is  $G$ -simple and that  $Z(A \rtimes_\sigma G)$  is a field. Let  $I$  be a non-zero ideal of  $A \rtimes_\sigma G$ . By Proposition 4.2 we conclude that  $I \cap Z(A \rtimes_\sigma G) \neq \{0\}$ . Hence  $1 \in I$  and therefore  $I = A \rtimes_\sigma G$ . This shows that  $A \rtimes_\sigma G$  is simple.

(d) We need to show that (iii) implies (i). The rest follows from (a) and (c).

Suppose that  $A$  is  $G$ -simple and that  $\sigma$  is injective. By Proposition 5.1  $A$  is a maximal commutative subring of  $A \rtimes_\sigma G$  and hence, by Theorem 1.1,  $A \rtimes_\sigma G$  is simple.  $\square$

*Proof of Theorem 1.3.* (a) It follows from Theorem 1.1 that (i) and (ii) are equivalent. The other claim follows from Theorem 1.2(a), Lemma 6.1 and Lemma 6.2.

(b) This follows immediately from Lemma 6.1 and Lemma 6.2.

(c) Consider Example 6.1. In the example it is explained that (iii), (iv) and (v) hold, but that (i) and (ii) fail to hold.

(d) This follows from (a), (b) and Theorem 1.2(d).  $\square$

**Remark 5.** If  $A$  is commutative, then injectivity of  $\sigma$  clearly implies outerness of  $G$ . Hence, an alternative way of proving that (iii) implies (i) in Theorem 1.2 (under the assumption that  $A$  is commutative and  $G$  is abelian), is by applying Corollary 4.4.

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## References

- [1] Auslander, M., *Rational singularities and almost split sequences*. Trans. Amer. Math. Soc. **293** (1986), no. 2, 511–531.

## References

- [2] Auslander, M., Reiten, I., *Grothendieck groups of algebras and orders*. J. Pure Appl. Algebra **39** (1986), no. 1-2, 1–51.
- [3] Auslander, M., Reiten, I., *Almost split sequences for rational double points*. Trans. Amer. Math. Soc. **302** (1987), no. 1, 87–97.
- [4] Crow, K., *von Neumann regular skew group rings*. Thesis (Ph.D.)-University of California, Santa Barbara, 2004.
- [5] Crow, K., *Simple regular skew group rings*, J. Algebra Appl. **4** (2005), no. 2, 127–137.
- [6] Demonet, L., *Skew group algebras of path algebras and preprojective algebras*. J. Algebra **323** (2010), no. 4, 1052–1059.
- [7] Fisher, J. W., Montgomery, S., *Semiprime skew group rings*. J. Algebra **52** (1978), no. 1, 241–247.
- [8] Hartwig, J. T., Öinert, J., *Simplicity and maximal commutative subalgebras of twisted generalized Weyl algebras*. Preprint available at arXiv:1009.4892v2 [math.RA]
- [9] Hou, B., Yang, S., *Skew group algebras of deformed preprojective algebras*. J. Algebra **332** (2011), 209–228.
- [10] de Jeu, M., Svensson, C., Tomiyama, J., *On the Banach  $*$ -algebra crossed product associated with a topological dynamical system*. To appear in Journal of Functional Analysis, 2012. Preprint available at arXiv:0902.0690v2 [math.OA]
- [11] Mihovski, S. V.,  *$A$ -invariant ideals of crossed products*. Comm. Algebra **29** (2001), no. 8, 3507–3522.
- [12] Montgomery, S., *Fixed rings of finite automorphism groups of associative rings*, Lecture Notes in Mathematics, 818. Springer, Berlin, 1980.
- [13] von Neumann, J., *Collected works. Vol. III: Rings of operators*. Pergamon Press, New York-Oxford-London-Paris, 1961.
- [14] Öinert, J., *Simple group graded rings and maximal commutativity*, Operator Structures and Dynamical Systems (Leiden, NL, 2008), 159–175, Contemp. Math. **503**, Amer. Math. Soc., Providence, RI, (2009).
- [15] Öinert, J., *Ideals and Maximal Commutative Subrings of Graded Rings*, xii+171 pp., Doctoral Thesis in Mathematical Sciences 2009:5, ISSN-1404-0034, LUTFMA-1038-2009, ISBN 978-91-628-7832-0, Lund University, 2009.
- [16] Öinert, J., Lundström, P., *The Ideal Intersection Property for Groupoid Graded Rings*. To appear in Communications in Algebra, 2012. Preprint available at arXiv:1001.0303v3 [math.RA]

## References

- [17] Osterburg, J., Passman, D. S., *What makes a skew group ring prime?* Azumaya algebras, actions, and modules (Bloomington, IN, 1990), 165–177, Contemp. Math., 124, Amer. Math. Soc., Providence, RI, 1992.
- [18] Osterburg, J., Peligrad, C., *A strong Connes spectrum for finite group actions of simple rings.* J. Algebra **142** (1991), no. 2, 424–434.
- [19] Passman, D. S., *Infinite crossed products.* Pure and Applied Mathematics, 135. Academic Press, Inc., Boston, MA, 1989.
- [20] Power, S. C., *Simplicity of  $C^*$ -algebras of minimal dynamical systems,* J. London Math. Soc. (2), **18** (1978), 534–538.
- [21] Reiten, I., Riedtmann, C., *Skew group algebras in the representation theory of Artin algebras.* J. Algebra **92** (1985), no. 1, 224–282.
- [22] Williams, D. P., *Crossed products of  $C^*$ -algebras.* Mathematical Surveys and Monographs, 134. American Mathematical Society, Providence, RI, 2007.
- [23] Wu, Q.-S., Zhu, C., *Skew group algebras of Calabi-Yau algebras.* J. Algebra **340** (2011), 53–76.
- [24] Xu, Y., Shum, K. P., *Skew group rings and Galois theory for skew fields.* Int. J. Contemp. Math. Sci. **3** (2008), no. 1-4, 49–62.
- [25] Yamskulna, G., *The relationship between skew group algebras and orbifold theory.* J. Algebra **256** (2002), no. 2, 502–517.
- [26] Zalesskii, A. E., Neroslavskii, O. M., *Simple Noetherian rings* (Russian). Vesci Akad. Navuk BSSR Ser. Fiz.-Mat. Navuk, **138** (1975), no. 5, 38–42.